Short Papers

Electrostatics of the Microstrip—Revisited

P. SILVESTER AND PETER BENEDEK

Abstract—**The well-known integral-equation formulation of the microstrip problem is solved by a projective method using trial functions that preserve the essential singularity in charge distribution at the strip edges. Suitable computer programs are presented. This formulation is believed particularly useful in the analysis of strip discontinuities, where details of the charge distribution cannot easily be traded off against speed of computation.**

Introduction

Numerous solutions have appeared in the literature for the electrostatic capacitance of microstrip transmission lines based on conformal mapping [1], substrip approximations to the integral-equation formulation of the problem [2], [3], variational formulations [4], as well as others. All these methods have produced quite good approximations to the capacitance values for strips both wide and narrow, as one might well expect, since the electrostatic capacitance is variationally stationary. Hence, even relatively large errors in the computed charge distributions will yield acceptably good values of C. As the analysis of discontinuity effects (open circuits, bends, and others) assumes greater importance, however, an accurate knowledge of the charge distribution itself becomes increasingly necessary. Of the published methods for finding charge distribution, only the substrip approximations [2], [3] can be expected to furnish reasonably good results; polynomial approximations cannot do so because of the excessively smooth behavior of polynomials near the strip edges. A method is therefore desired that will have accuracy at least comparable to the substrip solutions, but which will not consume large amounts of computing time. Below, a method is proposed that is capable of dealing with the electrostatics of both single and coupled strips, which takes little computing time but yields very good charge-distribution accuracy, including preserving the all-important singularity at the strip edge.

Formulation of the Problem

As in previous work, the TEM formulation of the microstrip problem is used. The integral equation that governs the electrostatic charge distribution on the strip, with reference to Fig. 1, is [2]

$$\phi(x) = \int_{-1}^{+1} \sigma(\xi)G(x;\xi)\,d\xi \tag{1}$$

where ϕ is the electric potential on the strip, σ is the charge distribution, and $G(x;\xi)$ is the Green's function for the problem. It can be shown, using extended image theory, that the necessary Green's function is

$$G(x;\xi) = \frac{1}{2\pi(\epsilon_0+\epsilon_1)}\sum_{n=1}^{\infty} K^{n-1}\log\frac{4n^2+\left(\frac{x-\xi}{h}\right)^2}{4(n-1)^2+\left(\frac{x-\xi}{h}\right)^2} \tag{2}$$

where $K=(\epsilon_0-\epsilon_1)/(\epsilon_0+\epsilon_1)$. It will be noted that the Green's function contains a singularity of the form $\log|x-\xi|$.

As is well known [5], the charge distribution σ on the strip is continuous, with singularities at the strip edges. It will be assumed to be of the form

$$\sigma(\xi) = \frac{c(\xi)}{\sqrt{1-\xi^2}} \tag{3}$$

where $c(\xi)$ is a slowly varying continuous function. According to the Weierstrass approximation theorem, such functions are well approximated by polynomials. Therefore, a good family of functions for approximating the charge distribution will be $\{\psi_n\}$, given by

$$\psi_n(x) = \frac{f_n(x)}{\sqrt{1-x^2}} \tag{4}$$

where

$$f_n(x) = \prod_{i=1}^{n-1}\left[\left(\frac{i}{n-1}\right)^2 - x^2\right], \qquad n>1.$$
$$f_1(x) = 1. \tag{5}$$

Approximating the charge distribution by

$$\sigma(\xi) = \sum_{i=1}^{k} a_i\psi_i(\xi) \tag{6}$$

the integral equation (1) assumes the form

$$\phi(x) = \sum_{i=1}^{k} a_i \int_{-1}^{+1}\psi_i(\xi)G(x;\xi)\,d\xi. \tag{7}$$

To solve for the coefficients a_i, one variant of the Galerkin–Petrov method [6] will be used. Projecting both sides of (7) onto a finite set of even-order Legendre polynomials $P_{2j}(x)$, one obtains

$$\int_{-1}^{+1}\phi(x)P_{2j}(x)\,dx = \sum_{i=1}^{k} a_i\int_{-1}^{+1}\int_{-1}^{+1}\psi_i(\xi)P_{2j}(x)G(x;\xi)\,d\xi dx. \tag{8}$$

It might be noted in passing that the integral projection in (8) cannot be regarded as a moment method [7]—not all members of the set $\{\psi_n\}$ are square-integrable, and therefore do not belong to any normed space on which the product integral constitutes an inner product. Nevertheless, (8) may be regarded as a matrix equation, which may be solved readily for the coefficients a_i.

No difficulty attaches to forming the integrals on the left side of (8); in fact, for a microstrip of constant potential, all Legendre polynomials except $P_0(x)$ are orthogonal to the potential function. However, the double integral on the right contains a singular kernel. Its evaluation may therefore cause some concern. Fortunately, the integral can be shown to be convergent, so that it may be evaluated readily, provided suitable weighted quadrature formulas are available. Such formulas may be constructed in the manner indicated by Silvester and Hsieh [8]; alternatively, suitable product quadrature rules may be obtained using the program described by Gautschi [9], or any equivalent program.

Working Program INFSTR

The above analysis has been incorporated in a Fortran program useful for calculating charge distributions expressed in series up to five terms. For economic evaluation of the singular integrals of (8), a quadrature rule of the form

$$\int\int f(x)g(y)\frac{\log|x-y|}{\sqrt{1-x^2}}dxdy = \sum_{ij} A_{ij}f(x_i)g(y_j) \tag{9}$$

is desired. Such formulas were developed for the final program version by taking Gaussian quadrature formulas with weight $(1-x^2)^{-1/2}$, as given by Stroud and Secrest [10], and adjoining Gaussian quadrature formulas specially computed with the weight function $\log[|x-\xi_n|/(|x-\xi_n|+1)]$, where ξ_n represents nodes of the first quadrature formula. This weight function has an essential singularity similar to that of the Green's function of the microstrip problem, but it does not change sign within the interval of integration. Ten-point quadratures in both directions have been found adequate to give good accuracy in microstrip problems where the width-to-height ratio of the strip does not exceed 3.0. For wider strips, the formulation appears to be entirely adequate, but the ten-point quadratures no longer suffice for accurate projections.

It is worth noting that the approximation involved in (4), (5) is in fact an approximation in polynomials with a Chebyshev weight. Since any polynomials of given degree span exactly the same function space, they may readily be converted to another family of

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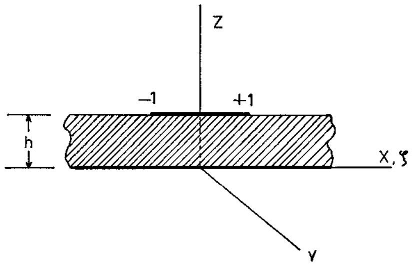

Fig. 1. Cross section of microstrip line.

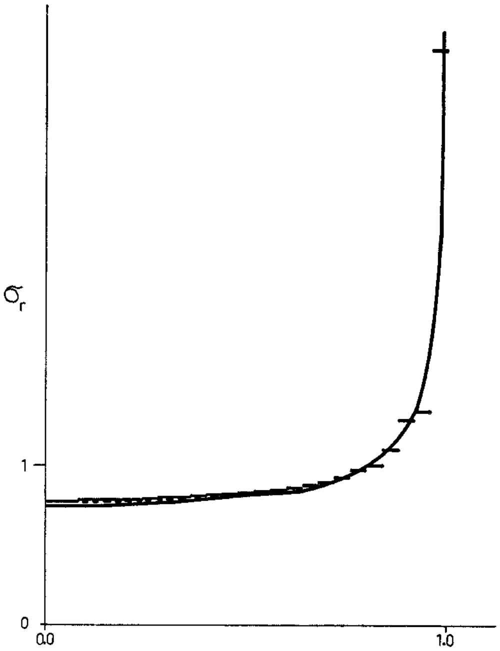

Fig. 2. Charge distributions obtained for microstrip line, five substrate thicknesses wide, using three methods of computation. The solid line exhibits the results from INFSTR using a two-term expansion; the dotted line shows the exact result obtained by conformal mapping. The staircase curve is that given by MICRO The relative permittivity of the substrate is assumed to be unity, i.e., there is no physical substrate.

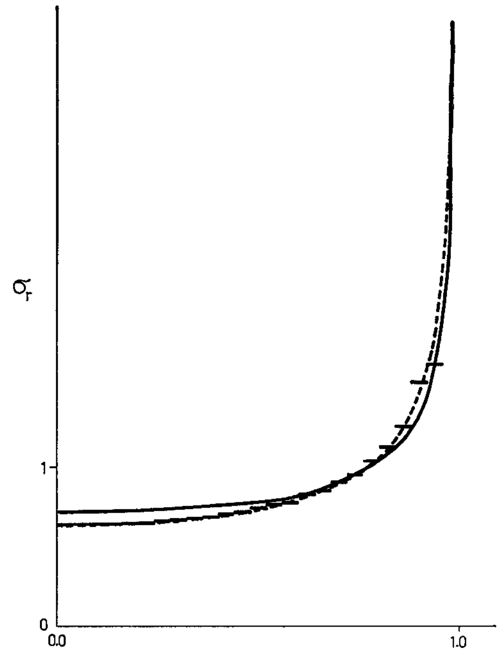

Fig. 3. Similar to Fig. 2, but for a microstrip line 0.1 substrate thicknesses wide.

TABLE I

w/h	ϵ_r	Substrip Method (30×30 matrix)	This Method (2×2 matrix)
0.2	2.5	28.2	28.6
2.667	2.5	92.2	92.5
0.2	4.2	42.9	43.5
2.667	4.2	145.9	146.0
0.2	9.0	84.1	85.4
2.667	9.0	296.8	296.5
0.2	16.0	144.2	146.4
2.667	16.0	516.5	515.7
0.2	51.0	444.6	451.3
2.667	51.0	1614.6	1611.0

polynomials not exceeding the same degree. It will be appreciated readily that this approximation is in fact equivalent to an approximation in Chebyshev polynomials, with the equal-ripple properties of the latter.

The program, in subroutine form, is believed sufficiently well documented to be usable by persons other than the authors. Written in a standard version of Fortran, it should be portable to almost any computing installation [12].

Applications to Striplines

Unfortunately, no exact results—that is to say, results of known superior accuracy—are known for strips on substrates of high permittivity. On the other hand, for parallel strips in free air, Palmer [11] has presented a detailed analysis, by means of conformal mapping, which permits computation of the capacitance to arbitrary accuracy. The analysis given by Palmer is sufficiently complicated virtually to preclude finding analytic expressions for the charge density. On the other hand, the positions of successive flux lines on the strips themselves may be determined from Palmer's analysis. Since these positions are known to a high accuracy, it is possible to perform numerical differentiations so as to plot the charge density on the strip surfaces. Programs to do so have been written so as to permit comparison with the results obtained from the program described above.

Fig. 1 shows comparative results obtained by conformal mapping and by INFSTR for a strip 5 times as wide as its height above ground plane, *in vacuo*. The charge distribution, it will be noted, is very similar for both the conformal mapping solution and the numerical approximation; however, the average charge densities differ sufficiently to lead to a capacitance error of under 2 percent. The essential feature important for analyses involving charge-distribution details, however, is obvious: the singularities at the edges are modeled much better than is the case with the substrip approximation also shown in Fig. 2. Similar comments apply to the results shown in Fig. 3 for a width-to-height ratio of 0.1: a quite narrow strip. In both cases, the approximations shown involve only two terms in the expansion, thus producing computing times lower than the substrip approximation. Computation on an IBM 360/75 is 0.7 s for the 2×2 matrix, as opposed to 1.8 s for the substrip method using a 25×25 matrix. Table I shows comparative values of microstrip capacitance by this method and by the substrip method.

Conclusions

The program described above, designed to permit accurate modeling of charge-density distributions on microstrip, is believed to be much more economic than earlier electrostatic approximation programs for determining the wave-propagation characteristics from microstrip. Their particular importance, however, will probably come to the fore in the detailed analysis of microstrip discontinuities, where the comparatively crude results obtained from polynomial approximations containing only a few terms, or from substrip approximations, appear not altogether adequate. When propagating-wave characteristics of microstrip are desired, the authors believe that the MICRO program [2] is the correct one to use for extremely wide str ps, while the new program described here is the correct one for relatively narrow strips. The limitation, while not serious, arises primarily in the method of numerical integration employed; that is to say, it is a program limitation rather than a limitation in the method, and may be removed, if desired, by using quadrature formulas of higher precision.

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Analysis of the Power Loss in the Coupling Mechanism of a Cavity Resonator

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***Abstract*—An analysis of a lossy coupling mechanism is presented. It is found that the reciprocal internal Q factor is augmented by a contribution $(1/2)\zeta(1/Q_{ex})$, where ζ depends on the parameters of the coupling mechanism and Q_{ex} is the external Q factor. The theory is applied to analyze the coupling to a superconducting high-Q cavity.**

Measurements on superconducting cavities with very high Q values have shown that coupling through a lossy coupling mechanism offers a special problem for obtaining the highest theoretical Q. Sucher and Fox [1] and Halbritter *et al.* [2] reported a decrease in the measured internal quality factor Q_0 of a cavity due to a lossy coupling mechanism. In the work on superconducting cavities, the author has analyzed the influence of the losses in a lossy coupling mechanism on the measured Q_0 of the cavity.

The analysis is performed as an analysis of the measurement of the internal Q_0 of the equivalent GCL circuit of the cavity through the equivalent lossy two-port of the coupling mechanism (Fig. 1).

The coupling factor at the input to the cavity is defined by

$$r = \frac{Q_0}{Q_{ex}} \tag{1}$$

where Q_{ex} is the external Q of the cavity seen from the two-port. The reflection coefficient at the input of the cavity, II, is then given by

$$\frac{a_2}{b_2} = \Gamma = \frac{r - 1 - jx}{r + 1 + jx} \tag{2}$$

where

$$x = \frac{2\Delta\omega}{\omega_0} Q_0 \tag{3}$$

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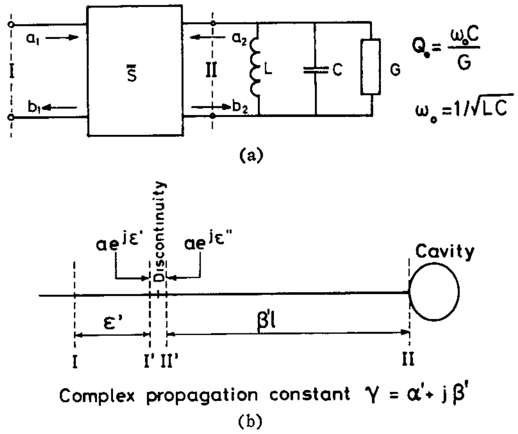

Fig. 1. (a) Equivalent circuit of cavity and coupling two-port. (b) Cavity with transmission-line coupling two-port.

and

$$\Delta\omega = \omega - \omega_0 \tag{4}$$

is the deviation of the frequency from the resonant frequency.

The scattering matrix of the two-port is

$$\bar{S} = \begin{Bmatrix} a & j\sqrt{b}e^{j\theta} \\ j\sqrt{b}e^{j\theta} & ce^{j\phi} \end{Bmatrix} \tag{5}$$

where a is real, if the reference plane at the input is selected suitably. The determinant is given by

$$\det \bar{S} = \Delta e^{j\eta} = ace^{j\phi} + be^{2j\theta} \tag{6}$$

where Δ is the magnitude and η the phase of the determinant.

The reflection coefficient squared at the input of the two-port, I, is found to be

$$|\Gamma'|^2 = |\Gamma_\infty'|^2 \frac{(x-\alpha)^2 + A^2}{(x-\beta)^2 + B^2} \tag{7}$$

where

$$|\Gamma_\infty'|^2 = \frac{\Delta^2 + 2\Delta a \cos\eta + a^2}{1 + 2c\cos\phi + c^2} \tag{8}$$

$$\alpha = \frac{2\Delta a \sin\eta}{\Delta^2 + 2\Delta a\cos\eta + a^2} r \tag{9}$$

$$\beta = \frac{2c\sin\phi}{1 + 2c\cos\phi + c^2} r \tag{10}$$

$$A = \left| r \frac{\Delta^2 - a^2}{\Delta^2 + 2a\Delta\cos\eta + a^2} - 1 \right| \tag{11}$$

$$B = \left| r \frac{1 - c^2}{1 + 2c\cos\phi + c^2} + 1 \right|. \tag{12}$$

It is seen that when $\alpha \neq \beta$, the resonance is unsymmetric and can have both a maximum and a minimum. In most conventional cavities the unsymmetry is not seen because the phase shift through the two-port is negligible, i.e., $\phi = \eta = 0$. It should be possible to determine all the involved parameters by measuring $|\Gamma'|^2$ as a function of the frequency. However, for high-Q cavities it is normally difficult to measure the response with sufficient accuracy because of the difficulty in accurately determining the frequency. Further, from an application point of view, e.g., application to narrow-band filters and frequency stabilization, it is desirable to obtain a resonance curve with a small bandwidth. Therefore, the subsequent treatment is concentrated on the derivation of the internal Q expressed by the measured internal Q at the input of the two-port.

It is assumed that the two-port parameters are frequency independent over the bandwidth of the resonator. $|\Gamma'|^2$ has a minimum value $|\Gamma'|_m^2$ at resonance, $x = x_m$. In the following derivations it is assumed that $|x_m - \beta| \ll B$ and $A < B$. It is then found that

$$x_m = \frac{B^2\alpha - A^2\beta}{B^2 - A^2} \tag{13}$$